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THE INVERSE ELECTROMAGNETIC SCATTERING PROBLEM FOR A PERFECTLY CONDUCTING CYLINDER*

by

David Colton
Department of Mathematical Sciences
University of Delaware
Newark, Delaware 19711

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ABSTRACT

We consider the problem of determining the shape of the cross section of a simply connected perfectly conducting infinite cylinder from a knowledge of the far field pattern for all angles of observation and small values of the wave number. The method we propose relies heavily on conformal mapping techniques. In particular we show that module the transfinite diameter each Fourier coefficient of the far field pattern of the electric field determines a Laurent coefficient of the conformal mapping taking the exterior of the unit disk onto the exterior of the unknown cross section. The transfinite diameter is determined by changing the polarization of the incoming wave and measuring the far field pattern of the resulting magnetic field. Of particular interest is the case when only a finite number of the Fourier coefficients of the far field pattern are known, and in this situation we obtain error estimates by using results on coefficient estimates for univalent functions.

I. Introduction.

In this paper we shall survey and extend some recent results we have obtained on the two dimensional low frequency inverse scattering problem for electromagnetic waves which we feel offers a considerable improvement over previous work in the area. In particular we are able to (1) circumvent the need for numerical analytic continuation, (2) show how elementary polarization effects can be utilized, (3) reduce the original infinite dimensional problem to one of finite dimension with explicit estimates given on the error resulting from such a reduction, and finally (4) show how a priori geometric knowledge on the shape of the scattering obstacle can be used to improve these error estimates. These results are obtained by a heavy reliance on the use of conformal mapping methods and the theory of univalent functions and hence are basically restricted to the two dimensional inverse scattering problem. However it is hoped that the insight thus gained in the two dimensional case can show the way to further progress in the case of three dimensions (For the case of acoustic waves in three dimensions some steps in this direction can be found in [2] and [4]). Our basic result can be described as follows. Consider the problem of the scattering of a plane time harmonic electromagnetic wave by an infinite perfectly conducting cylinder. Then from a knowledge of the low frequency limit of the first N Fourier coefficients of the electric far field pattern polarized parallel to the

generator of the cylinder, together with a single Fourier coefficient of the magnetic far field pattern of a field polarized perpendicular to the one originally considered, it is possible to determine the first N+1 Laurent coefficients of the conformal mapping which maps the exterior of the unit disk onto the exterior of the cross section of the unknown cylinder. Hence an approximation to the shape of the cylinder can be obtained, and since the mapping function is univalent error estimates can be obtained by using the Area Theorem together with special results in the geometric theory of functions (c.f. [8]). We do not discuss the error due to the (finite dimensional) problem of computing the N+1 Fourier coefficients. Such errors are basically due to the problem of accurately measuring the phase and amplitude of the far field pattern along with the problem of determining the Fourier coefficients of the far field pattern when only part of the far field is known. For a discussion of certain aspects of these last two problems we refer the reader to [7] and [5] respectively.

II. Reduction of the Inverse Scattering Problem to a Nonlinear Moment Problem.

Let D be the (bounded) cross section of a simply connected perfectly conducting infinite cylinder with generator parallel to the z axis such that D contains the origin and the boundary ∂D of D is smooth. Let E_z be the z component of the total

electric field due to the scattering by the cylinder of an incident time harmonic plane wave with electric vector given by $\vec{E}_i = e^{ikx}\,\hat{e}_z$ where the time dependence factor $e^{-i\omega t}$ has been factored out, (x,y,z) are Cartesian coordinates, k is the wave number, and $\hat{e}_z$ is the unit vector in the z direction. Then E_z will be a solution of the scalar Helmholtz equation

$$\Delta_2 E_z + k^2 E_z = 0 \tag{1}$$

in $R^2\backslash\overline{D}$, vanishes on ∂D, and has the asymptotic behaviour

$$E_z(r,\theta) = e^{ikr\cos\theta} + \frac{1}{4}\,e^{i(kr+\pi/4)}\sqrt{\frac{2}{\pi kr}}\,E(\theta;k) + O(r^{-3/2}) \tag{2}$$

where (r,θ) are polar coordinates. $E(\theta;k)$ is known as the far field pattern and our aim is to determine the shape of D from a knowledge of $E(\theta;k)$ for small values of the wave number k and $0 \le \theta \le 2\pi$. In particular it follows from Green's formula and the asymptotic behaviour of Hankel's function (c.f. [3]) that

$$E(\theta;k) = -\int_{\partial D} \frac{\partial E_z(\rho,\phi)}{\partial\nu}\exp[-ik\,\rho\cos(\theta-\phi)]\,ds \tag{3}$$

where ν denotes the unit outward normal to ∂D and (ρ,ϕ) are the polar coordinates of a point on ∂D. If we now expand $E(\theta;k)$ in a Fourier series we have

$$E(\theta;k) = \sum_{-\infty}^{\infty} a_n(k)\,e^{in\theta} \tag{4}$$

where

$$a_n(k) = -\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{\partial D}\frac{\partial E_z(\rho,\phi)}{\partial\nu}\exp[-in\theta-ik\rho\cos(\theta-\phi)]\,ds\,d\theta$$

$$= -i^{-n}\int_{\partial D}\frac{\partial E_z(\rho,\phi)}{\partial\nu}J_n(k\rho)\,e^{-in\phi}\,ds \tag{5}$$

and J_n denotes Bessel's function of order n.

In order to proceed further we need to determine the low frequency behaviour of $\partial E_z/\partial\nu$ on ∂D. Using the method of integral equations this was rigorously determined in [3] (see also [6]) with the result that for $(r,\theta) \in \partial D$

$$\frac{\partial E_z}{\partial\nu}(r,\theta) = \frac{1}{\log k}\frac{\partial E_o}{\partial\nu}(r,\theta) + 0\left(\frac{1}{(\log k)^2}\right) \tag{6}$$

where $E_o(r,\theta)$ is the solution of the following exterior Dirichlet problem for Laplace's equation:

$$E_o(r,\theta) = \log\frac{1}{r} + E_o^s(r,\theta) \quad \text{in } R^2\backslash\bar{D}$$

$$\Delta_2 E_o = o \quad \text{in } R^2\backslash\bar{D} \tag{7}$$

$$E_o = o \quad \text{on } \partial D$$

E_o^s is bounded as r tends to infinity.

It is possible to express $E_o(r,\theta)$ in terms of the (unique) conformal mapping $w=f(z)$ which maps the exterior of the domain D onto the exterior of the unit disk Ω such that at infinity $f(z)$ has the expansion

$$f(z) = a\,z + b + \frac{c}{z} + \frac{d}{z^2} + \ldots \qquad ; a > o. \tag{8}$$

Indeed it can be immediately verified that if $f(z)$ is as defined above, then

$$E_o(r,\theta) = -\log|f(z)| \qquad ; z = re^{i\theta}. \tag{9}$$

Hence from (5), (6), (9) and the Taylor series expansion of Bessel's function we have that for $n=0, 1, 2,\ldots$

$$a_n(k) = \frac{i^{-n}k^n}{2^n n!\log k} \int_{\partial D} \frac{\partial}{\partial\nu} \log|f(\rho e^{i\phi})|\rho^n e^{-in\phi}\, ds + 0\left(\frac{k^n}{(\log k)^2}\right). \tag{10}$$

Since $J_n(k\rho) = (-1)^n J_{-n}(k\rho)$ the same expression (up to conjugation and a factor of plus or minus one) holds for $n \le -1$ and thus no new information can be gathered from the negatively indexed Fourier coefficients. It can furthermore be easily verified that the $n=o$ term gives no information on the shape of D. Thus if we define (for $n \ge 1$)

$$\mu_n = i^{-n}\, 2^n\, n! \lim_{k\to o} \frac{\overline{a_n(k)}\,\log k}{k^n} \tag{11}$$

we arrive at the relation

$$\mu_n = \int_{\partial D} \frac{\partial}{\partial \nu} \log \mid f(\rho e^{i\phi}) \mid \rho^n e^{in\phi} \, ds. \tag{12}$$

We note that in practice only a small number of the μ_n can be computed with any degree of accuracy since for large n small errors in the measurement of the $a_n(k)$ will cause large errors in the μ_n due to the factor of $2^n n!$ in (11) (However see Section III). Hence we shall assume that the μ_n are known only for $n=1,2,\dots,N+1$ where N is a (small) positive integer; however as discussed in the Introduction we shall assume for the purposes of exposition that these $N+1$ numbers are known exactly.

As it stands, (12) is not too much help, since what is needed is a relationship between μ_n and $f^{-1}(w)$ since then by determining $f^{-1}(w)$ and evaluating it on the unit circle $|w|=1$ we can determine points on ∂D. However from the Cauchy-Riemann equations we have

$$\frac{\partial}{\partial \nu} \log \mid f(\rho e^{i\phi}) \mid = -\frac{\partial}{\partial s} \arg f(\rho e^{i\phi}) \tag{13}$$

and hence from (12) we have

$$\mu_n = -\int_{\partial D} \frac{\partial}{\partial s} \arg f(\rho e^{i\phi})\ \rho^n e^{in\phi}\ ds$$

$$= -\int_{|w|=1} \frac{\partial \arg w}{\partial w} [f^{-1}(w)]^n\ dw \qquad (14)$$

$$= i\int_{|w|=1} \frac{1}{w} [f^{-1}(w)]^n\ dw$$

for $n=1,2,\ldots,N+1$. (14) is the nonlinear "moment" problem of the title of this section.

The relationship (14) is unfortunately not sufficient to determine an approximation to $f^{-1}(w)$ for $|w|=1$. Indeed, as will be seen on the sequel, (14) only allows us to compute the first $N+1$ Laurent coefficients of $f^{-1}(w)$ module the coefficient a appearing in (8). The number a has geometric significance in that it can be shown (c.f. [8]) that a^{-1} is the <u>transfinite</u> <u>diameter</u> of the domain D. In order to determine this quantity we need to augment the relation (14) and this is accomplished by transmitting a second plane wave that is polarized perpendicular to the one originally sent. In particular let H_z be the z component of the total magnetic field due to the scattering by the cylinder of an incident time harmonic plane wave with magnetic vector given by $\vec{H}_i = e^{ikx}\,\hat{e}_z$, where we are

using our previously defined notation and the time dependent factor has been suppressed. Then H_z will be a solution of the scalar Helmholtz equation in $R^2\backslash\bar{D}$ with vanishing normal derivative on ∂D and having the asymptotic behaviour

$$H_z(r,\theta) = e^{ikr\cos\theta} + \frac{1}{4} e^{i(kr+\pi/4)} \sqrt{\frac{2}{\pi kr}} \quad H(\theta;k) + O(r^{-3/2}). \tag{15}$$

As is the case of the electric far field pattern we assume that the far field pattern $H(\theta;k)$ is known for small values of the wave number k and $0 \le \theta \le 2\pi$. We can then write ([1])

$$H(\theta;k) = \int_{\partial D} H_z(\rho,\phi) \frac{\partial}{\partial\nu} \exp\left[-ik\rho\cos(\theta-\phi)\right] ds \tag{16}$$

and if we expand $H(\theta;k)$ in a Fourier series

$$H(\theta;k) = \sum_{-\infty}^{\infty} b_n(k)\, e^{in\theta} \tag{17}$$

we find that ([1])

$$b_n(k) = -i^{-n} \int_{\partial D} H_z(\rho,\phi) \frac{\partial}{\partial\nu} \left[J_n(k\rho)\, e^{-in\phi}\right] ds. \tag{18}$$

We now note that ([1]) for $(r,\theta) \in \partial D$

$$H_z(r,\theta) = 1+ik\, H_o(r,\theta) + O(k^2 \log k) \tag{19}$$

where $H_o(r,\theta)$ is the solution of

$$H_o(r,\theta) = r\cos\theta + H_o^s(r,\theta) \quad \text{in } R^2\backslash\overline{D}$$

$$\Delta_2 H_o = 0 \text{ in } R^2\backslash\overline{D} \tag{20}$$

$$\frac{\partial H_o}{\partial \nu} = 0 \text{ on } \partial D$$

H_o^s is regular at infinity,

i.e. in terms of the conformal mapping (8) we can write

$$H_o(r,\theta) = \frac{1}{a} \operatorname{Re} \left[f(z) + \frac{1}{f(z)}\right]. \tag{21}$$

We can now proceed exactly as we did in the case when the electric vector was parallel to the generator of the cylinder. In particular if we define γ_o and γ_1 by

$$\gamma_o = -\lim_{k\to o} \frac{b_o(k)}{k^2}$$

$$\gamma_1 = \lim_{k\to o} \frac{\overline{b_1(k)}}{k^2} \tag{22}$$

we find from (18), (19), (21) that

$$\gamma_o = \text{Area of } D \tag{23}$$

$$\gamma_1 = \frac{i}{2a} \oint_{|w|=1} \left[1 - \frac{1}{w^2}\right] f^{-1}(w)\, dw. \tag{24}$$

We now want to use the relations (14), (23), (24) to determine an approximation to $f^{-1}(w)$ on $|w|=1$ (and hence an approximation to ∂D) together with error estimates under the assumption that the numbers $\mu_1, \mu_2, \ldots, \mu_{N+1}, \gamma_o, \gamma_1$ are known. This will be the topic of the following section.

III. Determination of an Approximation to ∂D and Error Estimates.

We now consider the nonlinear moment problem (14) and the relations (23), (24), and note that there are two main sources of error in trying to use these relations to determine $f^{-1}(w)$ on $|w|=1$. The first is that the μ_n are determined by means of the limiting relation (11) and since the $a_n(k)$ are only known approximately this can lead to severe computational difficulties. The second source of error is more basic in that in practice it is only possible to compute a finite number of the $\mu_n, n=1,2,\ldots,N+1$, where in general N is small, and from these $N+1$ numbers together with γ_o and γ_1 we want to determine the function $f^{-1}(w)$, i.e. we clearly have insufficient information to determine $f^{-1}(w)$ exactly. Although we shall primarily be concerned with this second source of error, we pause briefly to offer a few comments on how to reduce the error in computing the μ_n. To this end suppose it is known "a priori" that D is contained in a ball of radius r_o and that $E_z(r,\theta)$ can be measured for

$r = r_o$, $0 \le \theta \le 2\pi$. Then expanding $E_z(r_o,\theta)-e^{ikr_o\cos\theta}$ in a Fourier series we have that for $r \ge r_o$, $0 \le \theta \le 2\pi$,

$$E_z(r,\theta) = e^{ikr\cos\theta} + \sum_{-\infty}^{\infty} \alpha_n(k) \frac{H_n^{(1)}(kr)}{H_n^{(1)}(kr_o)} e^{in\theta}, \tag{25}$$

where $H_n^{(1)}(kr)$ denotes a Hankel function of the first kind. If we now make use of the asymptotic behaviour of Hankel's function we find that the Fourier coefficients $\alpha_n(k)$ of $E_z(r_o,\theta)-e^{ikr_o\cos\theta}$ are related to the Fourier coefficients $\alpha_n(k)$ of $E(\theta;k)$ by

$$(-i)^{n+1} \frac{\alpha_n(k)}{H_n^{(1)}(kr_o)} = \frac{1}{4} a_n(k). \tag{26}$$

Therefore from (11) and (26) we have that for $n \ge 1$

$$\begin{aligned} \mu_n &= \lim_{k\to 0} \frac{i\ 2^{n+2}\ n!\ \overline{\alpha_n(k)}\ \log k}{k^n\ H_n^{(2)}(kr_o)} \\ &= 4\pi n\ r_o^{\ n} \lim_{k\to 0} \overline{\alpha_n(k)}\ \log k \end{aligned} \tag{27}$$

where $H_n^{(2)}(kr)$ denotes Hankel's function of the second kind, i.e.

$$\mu_n = 2n\ r_o^{\ n} \lim_{k\to 0} \log k \int_0^{2\pi} [\overline{E_z(r_o,\theta)} - e^{-ikr_o\cos\theta}]\ e^{in\theta}\ d\theta \tag{28}$$

for $n \ge 1$. In particular since

$$\int_0^{2\pi} \exp\,[-ikr\cos\theta+in\theta]\;d\theta = 2\pi i^n\, J_n(kr) \tag{29a}$$

$$E_z(r,\theta) = \frac{1}{\log k}\, E_o(r,\theta) + O\left(\frac{1}{(\log k)^2}\right), \tag{29b}$$

where $E_o(r,\theta)$ is the solution of (7), we have that

$$\mu_n = 2n\; r_o^{\,n} \int_0^{2\pi} \overline{E_o(r_o,\theta)}\; e^{in\theta}\; d\theta\;;\; n \geq 1 \tag{30}$$

and (29b), (30) provide a more practical method than (11) for computing the numbers μ_n.

We now turn to the problem of determining an approximation to $f^{-1}(w)$ on $|w|=1$ from the relations (14), (23), (24) under the assumption that γ_o, γ_1 and the μ_n are known exactly for $n=1,2,\ldots,N+1$. From (8) we have that $f^{-1}(w)$ has a Laurent expansion of the form

$$f^{-1}(w) = \frac{w}{a} + c_o + \frac{c_1}{w} + \frac{c_2}{w^2} + \ldots \tag{31}$$

and since ∂D is assumed to be smooth the series (31) is uniformly convergent on $|w|=1$. Hence from (14), (31) we have (computing the residue) that

$$\mu_1 = -2\pi c_o$$
$$\mu_2 = -2\pi(c_o^2 + \frac{2c_1}{a}) \tag{32}$$
$$\vdots$$

and in general

$$\mu_n = -2\pi n\ a^{-n+1}\ c_{n-1} + \text{lower order coefficients.} \tag{33}$$

Hence, module the transfinite diameter a^{-1}, we can determine the Laurent coefficients c_n recursively in terms of the numbers μ_n, n=1,2,... . To determine the transfinite diameter we use (24) and compute the residue to obtain

$$\gamma_1 = -\frac{\pi}{a}[c_1 - \frac{1}{a}] \tag{34}$$

and thus from (32), (34) we can determine a. Hence if we know $\mu_1, \mu_2, \ldots, \mu_{N+1}$ and γ_1 we can now determine $c_o, c_1, \ldots c_n$, i.e. the function

$$f_N^{-1}(w) = \frac{w}{a} + c_o + \frac{c_1}{w} + \ldots + \frac{c_N}{w^N} \tag{35}$$

An approximation to ∂D can be obtained if we evaluate $f_N^{-1}(w)$ on $|w|=1$ and the mean square error in this approximation is given by

$$E(f^{-1} - f_N^{-1}) = \frac{1}{2\pi}\int_0^{2\pi} |f^{-1}(e^{i\theta}) - f_N^{-1}(e^{i\theta})|^2 \, d\theta$$
$$= \sum_{n=N+1}^{\infty} |c_n|^2. \tag{36}$$

We now want to obtain an estimate for the magnitude of this error. From the Area Theorem in univalent function theory ([8]) we have, using (23), that

$$\gamma_o = \frac{\pi}{a^2} - \pi \sum_{n=1}^{\infty} n|c_n|^2. \tag{37}$$

Hence

$$E(f^{-1} - f_N^{-1}) \leq \frac{1}{N+1} \sum_{n=N+1}^{\infty} n|c_n|^2$$
$$\leq \frac{1}{N+1} [\frac{1}{a^2} - \frac{\gamma_o}{\pi}]. \tag{38}$$

An improvement on this error estimate can be obtained if it is known "a priori" that D is convex since in this situation we have ([8], p.50) that

$$|c_n| \leq \frac{2}{an(n+1)} \cdot \tag{39}$$

Hence in this case

$$
\begin{aligned}
E(f^{-1} - f_N^{-1}) &\le \frac{4}{a^2} \sum_{n=N+1}^{\infty} \frac{1}{n^2(n+1)^2} \\
&\le \frac{4}{a^2} \left(\frac{N+2}{N+1}\right)^2 \sum_{n=N+1}^{\infty} \frac{1}{(n+1)^4} \\
&\le \frac{4}{a^2} \left(\frac{N+2}{N+1}\right)^2 \int_{N+1}^{\infty} \frac{dx}{x^4} \\
&= \frac{4(N+2)^2}{3a^2(N+1)^5}
\end{aligned}
\tag{40}
$$

which is a considerable improvement on (38) except in the case when D is a small perturbation of a disk and we have $a^{-2} \approx \gamma_0 \pi^{-1}$.

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References

1. D. Colton, The inverse scattering problem for a cylinder, Proc. Royal Soc. Edinburgh, 84A (1979), 135-143.

2. D. Colton, Remarks on the inverse scattering problem for low frequency acoustic waves, J. Diff. Eqns., to appear.

3. D. Colton and R. E. Kleinman, The direct and inverse scattering problems for an arbitrary cylinder: Dirichlet boundary conditions, Proc. Royal Soc. Edinburgh, to appear.

4. D. Colton and R. Kress, Iterative methods for solving the exterior Dirichlet problem for the Helmholtz equation with applications to the inverse scattering problem for low frequency acoustic waves, J. Math. Anal. Appl., to appear.

5. W. A. Imbriale and R. Mittra, The two-dimensional inverse scattering problem, IEEE Trans. Antennas and Propagation A P-18 (1970), 633-642.

6. R. C. MacCamy, Low frequency acoustic oscillations, Quart. Appl. Math. 23 (1965), 247-256.

7. R. B. Mack, A. W. Wojcicki, and J. J. Andriotakis, An implementation of conventional methods of measuring the amplitude and phase of backscatter fields, Air Force Cambridge Research Laboratory Technical Report 73-0418, 1973.

8. C. Pommerenke, Univalent Functions, Vandenhoeck and Ruprecht, Göttingen, 1975.

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